Chapter H5

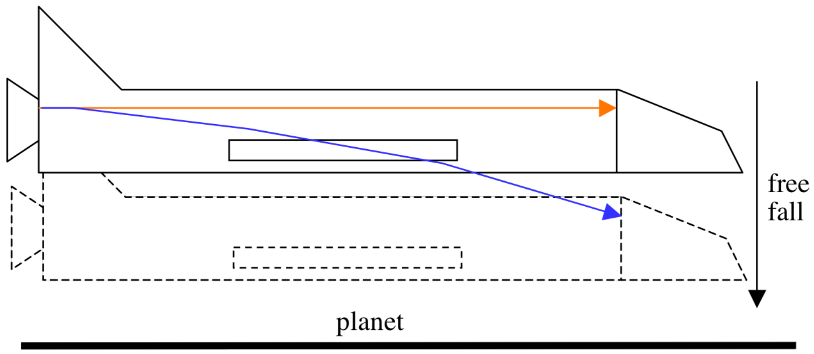
Q1 This is a true statement. Light follows the geodesics of spacetime. In the absence of matter, the spacetime is flat and the geodesics are the ordinary straight lines. In a curved spacetime, geodesics look bent because we are prejudiced into thinking in terms of flat spacetimes.

Q2 By the equivalence principle, this frame of reference is equivalent to one which is at rest in a gravitational field. The gravitational field strength is then directed to the left. The helium balloon will "rise" i.e. move opposite to the gravitational field. I.e. it will move to the right.

Q3 For exactly the same reasons as in problem 2 the flame will bend to the right.

Q4 The equivalence principle states that inertial effects (i.e. effects due to acceleration) cannot be distinguished from effects of gravitation. More precisely, it states that an accelerating frame of reference in outer space is equivalent to a frame of reference at rest in a uniform gravitational field whose field strength is the same as the acceleration of the other frame. It also states that a freely falling frame of reference in a gravitational field is equivalent to an inertial frame of reference.

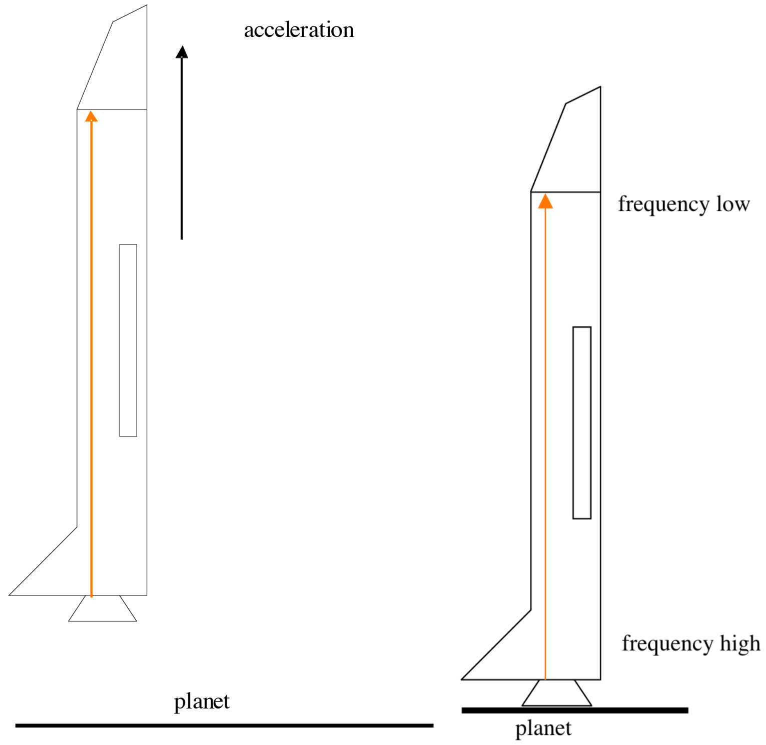
(a) Consider a rocket that is **freely falling** in a gravitational field. According to observers inside the rocket, the ray of light that is emitted from the back wall of the rocket will travel on a straight line and hit the front of the rocket at a point that has the same distance from the floor as the point of emission (path of light shown in the orange line). This is because to the occupants of the rocket, the rocket is equivalent to a truly inertial frame of reference.



But the rocket is seen to be falling by an observer outside. By the time the light ray goes across, the rocket has fallen and so the ray appears to be following the curved path shown in blue. Thus the outside observer claims that **in a gravitational field, light bends** towards the mass causing the field.

(b) Consider a rocket accelerating in outer space with acceleration a and one at rest in a gravitational field of strength $g = a$. The frames are equivalent. In both frames a spring of spring constant k is attached to the front of the rocket and a body is attached to the end of the spring. The spring must extend by the same amount in both cases. So $kx = m_g g$ for the rocket at rest and $kx = m_i a$ for the one accelerating. Hence $m_i = m_g$.

(c) The diagram on the left shows a rocket accelerating in outer space. The diagram to the right shows a rocket at rest on a massive body which is thus equivalent to the first frame. A ray of light is emitted from the back of the rocket and is received at the front.



To an observer outside the rocket on the left, the front of the rocket is moving away from the light ray and so there should be a Doppler redshift. I.e. the observer outside expects that the frequency of light measured at the reception point should be smaller than that at emission. Hence the outside observer must conclude that **as the ray of light moves higher in the gravitational field it suffers a redshift**. But frequency is the number of wavefronts received per second so how can the frequency change? The answer has to be that when one second goes by, according to a clock at the base, more than a second goes by, according to a clock at the top, i.e. the equivalence principle predicts **gravitational time dilation**: the interval of time between two events is longer when measured by a clock far from the gravitational field compared to a clock near the gravitational field.

Q5 As the radius gets smaller, a time will be reached when the radius of the object becomes equal to the Schwarzschild radius. The bending of space around the object will be substantial and the object will become a black hole.

Q6 The period of oscillation of the mass at the end of a spring is $T = 2\pi\sqrt{\frac{m}{k}}$. So the acceleration of gravity, i.e. the gravitational field strength does not enter. Hence the period will be the same in (a) and (b).

Q7 The emitted frequency is $f = \frac{3.00\times 10^{8}}{500.0\times 10^{-9}} = 6.00\times 10^{14}\text{ Hz}$. The shift is found from $\frac{\Delta f}{f} = \frac{gH}{c^2} \Rightarrow \Delta f = 6.00\times 10^{14}\times \frac{9.81\times 50.0}{(3.00\times 10^{8})^2} == 3.27\text{ Hz}$.

Q8 The frequency at emission is $f = \frac{3.00\times 10^{8}}{548\times 10^{-9}} = 5.47\times 10^{14}\text{ Hz}$. The Schwarzschild radius of the sun is $R_S = \frac{2\times 6.67\times 10^{-11}\times 2.0\times 10^{30}}{(3.0\times 10^{8})^2} = 2.96\times 10^{3}\text{ m}$ and its radius is $R = 6.96\times 10^{8}\text{ m}$. Hence, $\Delta f = 5.47\times 10^{14}\times \frac{2.96\times 10^{3}}{6.96\times 10^{8}} = 2.33\times 10^{9}\text{ Hz}$.

Q9 Because of time dilation at the position of the observer, the far away observer will measure (a) a longer wavelength because the frequency observed will be gravitationally redshifted i.e. be smaller, (b) a smaller frequency of reception between pulses and (c) a longer duration of the pulses. Quantitatively, the factor by which the shifts take place is given in problem 8, i.e. $\frac{R_S}{R} = \frac{1}{5} = 0.2$. Hence, $\lambda = 4.00\times 10^{-7}\times 1.2 = 4.8\times 10^{-7}\text{ m}$, duration between pulses is $1.00\times 1.2 = 1.2\text{ s}$ and duration is $1.00\times 1.2 = 1.2\text{ ms}$.

Q10 The acceleration experienced by the clock on the circumference will be greater. By the equivalence principle this clock will behave as an identical clock in a gravitational field. It will therefore run slow relative to a clock in a smaller gravitational field.

Q11 The mass of the earth is $M = 6.0\times 10^{24}\text{ kg}$. The Schwarzschild radius of the earth is $R = \frac{2GM}{c^2} = \frac{2\times 6.67\times 10^{-11}\times 6.0\times 10^{24}}{9.0\times 10^{16}} = 8.89\times 10^{-3}\text{ m}$. Its density would then be

$$\rho = \frac{M}{\frac{4\pi R^3}{3}} = \frac{6.0\times 10^{24}}{\frac{4\pi (8.89\times 10^{-3})^3}{3}} = 2.0\times 10^{30}\text{ kg m}^{-3}$$

. (This is larger than nuclear densities by factor of 10^{13}.)

Q12 From $R = \frac{2GM}{c^2}$, $R = \frac{2\times 6.67\times 10^{-11}\times 2.0\times 10^{31}}{(3.0\times 10^{8})^2} = 3.0\times 10^{4}\text{ m}$.

Q13 A geodesic is a curve in spacetime which has the least length compared to any other curve with the same beginning and end points.

Q14 In Newtonian mechanics the path would be explained by saying that a gravitational attractive force acts on the particle changing the original path in a curved path around the massive object. In relativity the path is explained by saying that the particle follows the geodesic in the curved spacetime around the massive object.

Q15 (a) From the point of view of an observer inside the spacecraft the ball will move on a straight line parallel to the floor and therefore hit the opposite wall at the same height as that at emission. From the point of view of an inertial observer outside with respect to whom the spacecraft moves upwards, the path will also be a straight line with an upward slope. (b) The situation is equivalent to a frame of reference in a gravitational field. Therefore the ball will curve towards the floor following a parabolic path and will hit the opposite wall lower.

Q16 (a) The frame of reference is an inertial frame and so the light ray will travel along a straight line hitting the opposite wall of the spacecraft at a point closer to the floor than at the point of entry. (b) The frame is now equivalent to a frame of reference at rest on the surface of a massive body. The ray of light will follow a bent path hitting the opposite wall of the spacecraft at a point closer to the floor than at the point of entry.

Q17 Looking at the figure of page 687 in the textbook, let θ be the angle at the center of the sphere subtended by the arc NP. Then $r = R\theta \Rightarrow \theta = \frac{r}{R}$. The radius of the circle whose circumference is $C(r)$, is then $R\sin\theta = R\sin\frac{r}{R}$. Hence $C(r) = 2\pi R\sin\frac{r}{R}$. The curvature is therefore

$$
\begin{aligned}
K &= 3\lim_{r\to 0}\frac{2\pi r - C(r)}{\pi r^3} \\
&= 3\lim_{r\to 0}\frac{2\pi r - 2\pi R\sin\frac{r}{R}}{\pi r^3}
\end{aligned}
$$

For small r, $\sin\frac{r}{R} \approx \frac{r}{R} - \frac{1}{3!}(\frac{r}{R})^3$ and so

$$
\begin{aligned}
K &= 3\lim_{r\to 0}\frac{2\pi r - 2\pi R(\frac{r}{R} - \frac{1}{6}\frac{r^3}{R^3})}{\pi r^3} \\
&= 3\lim_{r\to 0}\frac{2\pi r - 2\pi r + \frac{\pi}{3}\frac{r^3}{R^2}}{\pi r^3} \\
&= \lim_{r\to 0}\frac{\pi\frac{r^3}{R^2}}{\pi r^3} \\
&= \frac{1}{R^2}
\end{aligned}
$$

The area of the cap is

$$
\begin{aligned}
A &= \int_0^{r/R} C(r)\,dr \\
&= \int_0^{r/R} 2\pi R \sin\left(\frac{r}{R}\right) dr \\
&= -2\pi R^2 \cos\left(\frac{r}{R}\right)\Big|_0^{r/R} \\
&= 2\pi R^2 (1-\cos\frac{r}{R})
\end{aligned}
$$

For small r, $\cos\frac{r}{R} \approx 1 - \frac{1}{2!}(\frac{r}{R})^2 + \frac{1}{4!}(\frac{r}{R})^4$ and so

$$
\begin{aligned}
K &= 12\lim_{r\to 0}\frac{\pi r^2 - 2\pi R^2\left(1-(1-\frac{1}{2}\frac{r^2}{R^2}+\frac{1}{24}\frac{r^4}{R^4})\right)}{\pi r^4} \\
&= 12\lim_{r\to 0}\frac{\pi r^2 - \pi r^2 + \frac{\pi}{12}\frac{r^4}{R^2}}{\pi r^4} \\
&= \lim_{r\to 0}\frac{\frac{r^4}{R^2}}{r^4} \\
&= \frac{1}{R^2}
\end{aligned}
$$

Q18 (a) Because rays of light coming from low in the sky will be bent, these will never reach the observer. The observer sees that his horizon is rising and he can only see things within a vertical cone whose angle is decreasing. (b) Once inside the event horizon the observer can only see rays of light falling “vertically’ into the black hole, i.e. only along a single line.

Q19 The plane is flying at essentially a constant height and so on the surface of a sphere. This is curved and so the plane follows the geodesics of the sphere (these are great circles – circles whose plane goes the center of the earth) because these have the least length so the least amount of fuel is being used.

Q20 Einstein through the contraption into the air. Being in free fall, the brass ball is effectively weightless since, by the equivalence principle, it is equivalent to a ball in zero gravitational field. The spring will then pull the mass in the bowl. Einstein was very proud of his present and showed it to all who visited him.

Q21 We have that $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-\dfrac{R_S}{r}}}$, i.e. $2.0 = \dfrac{1.0}{\sqrt{1-\dfrac{R_S}{r}}}$ and so $1-\dfrac{R_S}{r} = \dfrac{1}{4}$, $\dfrac{R_S}{r} = \dfrac{3}{4}$ giving finally $r = \dfrac{4}{3}R_S$. The observer is at distance of $r = \dfrac{1}{3}R_S$ from the event horizon.

Q22 It will take longer since clocks near massive objects run slow compared to clocks far away. The accelerating spacecraft is equivalent to one at rest in a gravitational field.

Q23 (a) A ray of light appears to follow a bent path as it goes past the sun. This is because the space near the sun is curved. (b) A clock near a massive object runs slow compared to an identical clock far away. This is because time is bent.

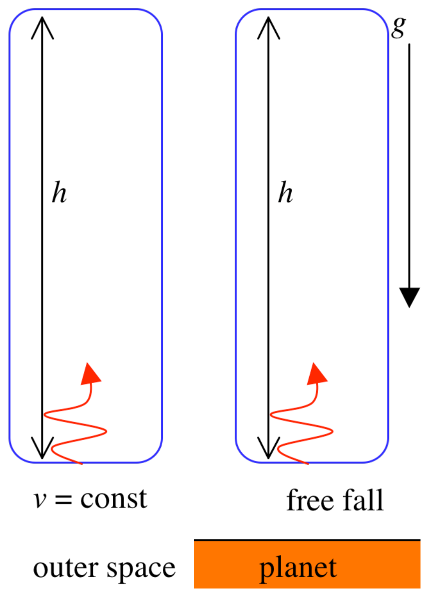
Q24 (a) A black hole is a singularity in spacetime, a point of infinite curvature. (b) $R = \dfrac{2GM}{c^2} = \dfrac{2\times 6.67\times 10^{-11}\times 5.0\times 10^{35}}{9.0\times 10^{16}} = 7.4\times 10^{8}\ \text{m}$. (c) This radius is the distance from the black hole where the escape speed is equal to the speed of light. The black hole does not have a radius since it is a point. (d) The observer next to the source measures a period of $T = \dfrac{1}{f} = \dfrac{1}{7.50\times 10^{14}} = 1.3\times 10^{-15}\ \text{s}$. (e) The distant observer will measure a period of $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-\dfrac{R_S}{r}}} = \dfrac{1.3\times 10^{-15}}{\sqrt{1-\dfrac{R_S}{1.1R_S}}} = 4.4\times 10^{-15}\ \text{s}$ and hence a frequency of $f = \dfrac{1}{4.4\times 10^{-15}} = 2.3\times 10^{14}\ \text{Hz}$.

25 (a) $R_S = \dfrac{2GM}{c^2}$. (b) The area is $A_S = 4\pi R_S^2 = \dfrac{16\pi G^2 M^2}{c^4}$. (c) Mass constantly falls into the black hole and so the radius keeps increasing. Hence the area also increases. (d) Entropy is another quantity that always increases. The black hole has thermodynamic properties (as discovered independently by D. Christodoulou and J. Bekenstein) and in fact behaves as a real thermodynamic black body that radiates according to the Stefan-Boltzmann law. This is because of quantum effects as explained by S. Hawking. The effective “temperature” of the black hole is inversely proportional to its mass. This means that small black holes radiate a lot.

26 (a) The ray will fall straight into the hole. (b) The rays will be bent and enter the observer’s eye.

Additional problem

A1 A box moves with constant velocity in outer space far from all masses. A photon is emitted from the base of the box and is received at the top. The frequency of the photon at emission, according to an observer in the box, is f_0.



(a) What is the frequency of the photon when it is received at the top according to observers inside the box?
(b) According to the equivalence principle, the situation is equivalent to a box that is freely falling in the gravitational field of a planet. Explain how an observer at rest on the planet reaches the same conclusion about the frequency of the received photon as the observer inside the box.

Answer (a) The photon is emitted in an inertial frame of reference and so the frequency at the top will be the same as that at emission, i.e. f_0. (b) According to the observer on the surface of the planet, the top is moving downwards and so the frequency will be Doppler shifted (a blueshift) such that $\frac{\Delta f}{f_0} = \frac{v}{c}$. Here v is the speed of the top of the box the instant it receives the photon. But $v = gt = g\frac{h}{c}$ since the photon takes a time $\frac{h}{c}$ to get to the top. Hence, $\frac{\Delta f}{f_0} = \frac{gh}{c^2}$. However, the photon has been emitted in a region of gravitational field and so as it climbs towards the top it will suffer a gravitational redshift equal to $\frac{\Delta f}{f_0} = \frac{gh}{c^2}$. The Doppler blueshift and the gravitational redshift are *exactly the same* and so cancel out leaving the frequency of the photon at the top f_0! Yes, the way the theory works is almost magical.